NATIONAL BUREAU OF STANDARDS REPORT

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ON INDEPENDENT SAMPLES FROM NORMAL POPULATIONS

by

A. A. Zinger

U. S. DEPARTMENT OF COMMERCE
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It is well known that for a sample from a normal population, the sample mean $\bar{x}$ and the sample variance s^2 are independent. This theorem was proven by R.A. Fisher. The question arises: To what extent does this property (the independence of $\bar{x}$ amd s^2) characterize a general population? Here we must point out the work of Lukacs where it is proven that if the general population has a moment of the second order, and if the sample mean and sample variance are independent in samples of size $n > 1$, then the general population is distributed normally. However, this theorem does not answer completely the stated question, since the condition demanding the existence of a general second moment imposes a considerable limitation. There remains the question of what the general distribution might be if we do not demand *a priori* the existence of a general second moment.

THEOREM. Let us consider n $(n>1)$ mutually independent random variables $x_1, x_2, \ldots, x_n$, each of which is distributed according to the law $F(x)$. Let us form $\bar{x} = (x_1+x_2+\ldots+x_n)/n$ and $s^2 = [(x_1-\bar{x})^2+\ldots+(x_n-\bar{x})^2]/n$. If $\bar{x}$ and s^2 are independent, then the distribution $F(x)$ is normal.

Proof. We note first the obvious algebraic identity

$$\sum_{j=1}^{n} x_j^2 = n\bar{x}^2 + ns^2 \tag{1}$$

and introduce the function $f(t)$:

$$f(t) = E(e^{itx-x^2}) \tag{2}$$

The symbol E here, and elsewhere, denotes the mathematical expectation of the random variable within the parentheses.

Let us note, that due to the finiteness of the integrals

$$\int_{-\infty}^{+\infty} x^n e^{-x^2} dF(x) \tag{3}$$

the function $f(t)$ is differentiable any number of times. To prove the theorem, it is necessary that $f(t)$ be differentiable only twice.

Examine the expression

$$E\left\{ns^2 \exp[\sum_{j=1}^{n}(itx_j-x_j^2)]\right\}. \tag{4}$$

Making use of the independence of $\bar{x}$ and s^2 and the identity (1), let us write

$$E\{ns^2\exp[\sum_{j=1}^{n}(itx_j-x_j^2)]\} = E(ns^2e^{-ns^2})E[\exp(it\sum_{1}^{n}x_i-n\bar{x}^2)] \tag{5}$$

Let us transform separately both sides of (5):

$$E(ns^2e^{-ns^2})E[\exp(it\sum_{j=1}^{n}x_j-n\bar{x}^2)] = \frac{E(ns^2e^{-ns^2})}{E(e^{-ns^2})}E[\exp(it\sum_{j=1}^{n}x_j-n\bar{x}^2)]E(e^{-ns^2}) \tag{6}$$

Again using the fact that $\bar{x}$ and s^2 are independent, we write

$$E(ns^2 e^{-ns^2})\, E[\exp(it \sum_{j=1}^{n} x_j - n\bar{x}^2)] = a E\{\exp[\sum_{j=1}^{n} (itx_j - x_j^2]\} \tag{7}$$

where we define

$$a = \frac{E(ns^2 e^{-ns^2})}{E(e^{-ns^2})} \,. \tag{8}$$

Furthermore, since all x_i are by definition mutually independent and since each is distributed by the law of $F(x)$,

$$E\{\exp[\sum_{j=1}^{n} (itx_j - x_j^2] = f^n(t) \tag{9}$$

Let us consider the transformation of the left side of equation (5). Let us substitute here from (1), $ns^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$

$$E\{ns^2 \exp[\sum_{j=1}^{n} (itx_j - x_j^2)]\} =$$

$$= E\{\sum_{j=1}^{n} x_j^2 \exp[\sum_{k=1}^{n} (itx_k - x_k^2)]\} - E\{n\bar{x}^2 \exp[\sum_{j=1}^{n} (itx_j - x_j^2)]$$

We make further the following transformation:

$$E\{ns^2 \exp[\sum_{j=1}^{n} (itx_j - x_j^2)]\} =$$

$$= nf^{n-1}(t)E[x^2 \exp(itx - x^2)] - \frac{1}{n} E\{(\sum_{j=1}^{n} x_j)^2 \exp[it \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j^2]\}$$

Let us evaluate $E[x^2 \exp(itx - x^2)]$ and $E\{(\sum_{j=1}^{n} x_j)^2 \exp[it \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j^2]\}$

$$\left.\begin{aligned} &E(x^2 e^{itx-x^2}) = -f''(t) \\ &E\{(\sum_{j=1}^{n} x_j)^2 \exp[it\sum_{j=1}^{n} x_j - \sum_{j=1}^{n} x_j^2]\} = -[f^n(t)]'' \end{aligned}\right\}$$

It follows that the identity (5) may be rewritten as follows

$$-nf^{n-1}(t)f''(t) + f^{n-1}(t)f''(t) + (n-1)f^{n-2}(t)[f'(t)]^2 = a\, f^n(t)$$

or

$$f^{n-1}(t)f''(t)-f^{n-2}(t)[f'(t)]^2 = -a_1 f^n(t), \qquad (a_1 = \frac{a}{n-1}) \tag{11}$$

Clearly $f(t)$ is a continuous function and $f(0) = \int e^{-x^2}dF) > 0$: therefore, in some neighborhood of zero $f(t) \neq 0$.

Let us solve equation (11) in this region. We divide both sides of the equation by $f^n(t)$, so as to obtain

$$\frac{f''(t)}{f(t)} - [\frac{f'(t)}{f(t)}]^2 = -a_1 \tag{12}$$

or $[\log f(t)]'' = -a_1$. The solution of this equation is

$$f(t) = k \exp[i\gamma t - \frac{a_1}{2} t^2]$$

Here

$$\left.\begin{aligned} k &= f(0) = \int e^{-x^2} dF(x) \\ \gamma &= \frac{1}{i} f'(0) = \int x e^{-x^2} dF(x) \end{aligned}\right\} \tag{13}$$

It is easy to show that this solution may be extended to the entire line. We shall make use of the fact that the set of zeros of $f(t)$ is closed. Assuming that $f(t)$ does have zeros, then there exists a value t_0 such that $f(t_0) = 0$, while for all values of t for which $|t|<|t_0|$, $f(t) \neq 0$. Since $f(t)$ is the solution of the equation it may be written in the form $k \exp[i\gamma t - \frac{a_1}{2}t^2]$. Let, for instance $t_0 > 0$. We can choose an ε sufficiently small, for which

$$f(t_0 - \varepsilon) \neq 0,$$

$$f(t_0 - \varepsilon) = k \exp[i\gamma(t_0 - \varepsilon) - \frac{a_1}{2}(t_0 - \varepsilon)^2]$$

From this it follows that $f(t_0) = \lim_{\varepsilon \to 0} f(t_0 - \varepsilon)$ cannot be equal to zero. Wehave thus shown that the solution $f(t) = k \exp[i\gamma t - \frac{a_1}{2} t^2]$ is true over the entire line.

Let us now investigate the function $\Omega(x)$, defined as follows:

$$\Omega(x) = \int_{-\infty}^{x} e^{-x^2} dF(x). \tag{14}$$

Then by definition $f(t)$ is the Fourier transform of the function $\Omega(x)$. But $f(t)$ is the characteristic function of the Gaussian law, and therefore $\Omega(x)$ may be represented as follows

$$\Omega(x) = \frac{k}{\sqrt{2\pi a_1}} \int_{-\infty}^{x} e^{-\frac{(x-\gamma)^2}{2a_1}} dx . \tag{15}$$

In other words, we may define the function $F(x)$ by the equation

$$\int_{-\infty}^{x} e^{-x^2} dF(x) = \frac{k}{\sqrt{2\pi a_1}} \int_{-\infty}^{x} e^{-\frac{(x-\gamma)}{2a_1}} dx. \qquad (16)$$

To solve this equation, let us note that for arbitrary x_1 and x_2,

$$\int_{x_1}^{x_2} e^{-x^2} dF(x) = \frac{k}{\sqrt{2\pi a_1}} \int_{x_1}^{x_2} e^{-\frac{(x-\gamma)^2}{2a_1}} dx \;. \qquad (17)$$

We shall prove that $F(x)$ is differentiable at every point, and we shall evaluate its derivative.

Let x_0 be some point, for example $x_0 > 0$, and let $h > 0$, then

$$\int_{x_0}^{x_0+h} e^{-x^2} dF(x) = \frac{k}{\sqrt{2\pi a_1}} \int_{x_0}^{x_0+h} e^{-\frac{(x-\gamma)^2}{a_1}} dx$$

and therefore

$$e^{-(x_0+h)^2} \int_{x_0}^{x_0+h} dF(x) < \int_{x_0}^{x_0+h} e^{-x^2} dF(x) < e^{-x_0^2} \int_{x_0}^{x_0+h} dF(x). \qquad (18)$$

From this it follows that

$$\frac{k}{\sqrt{2\pi a_1}} \frac{e^{x_0^2}}{h} \int_{x_0}^{x_0+h} e^{-\frac{(x-\gamma)^2}{2a_1}} dx < \frac{F(x_0+h)-F(x_0)}{h}$$

$$< \frac{k}{\sqrt{2\pi a_1}} \frac{e^{(x_0+h)^2}}{h} \int_{x_0}^{x_0+h} e^{-\frac{(x-\gamma)^2}{2a_1}} dx \;. \qquad (19)$$

An analogous inequality may be obtained for $h < 0$, and we thus obtain

$$F'(x_0) = \frac{k}{\sqrt{2\pi a_1}} \exp[x_0^2 - \frac{(x_0-\gamma)^2}{2a_1}] \tag{20}$$

The above theorem may be somewhat generalized as follows. Let $x_1, x_2, \ldots\ldots, x_n$ be mutually independent random variables, identically distributed according to the law $F(x)$. Let us form the statistics $\tilde{x} = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n$ (we assume that all α_k are non-negative, and that $\sum_{k=1}^{n} \alpha_k^2 = 1$)

and $s^2 = \sum_{k=1}^{n} x_k^2 - (\alpha_1 x_1 + \ldots \alpha_n x_n)^2$.

If $\tilde{x}$ and s^2 are independent, then the general population is normally distributed.

Proof. As in the previous theorem, let us introduce the function

$$f(t) = \int e^{itx-x^2} dF(x)$$

Then in the manner utilized in the previous theorem, we obtain the equation

$$\sum_{k=1}^{n} \frac{f''(\alpha_k t)}{f(\alpha_k t)} - \sum_{k=1}^{n} \alpha_i^2 \frac{f''(\alpha_k t)}{f(\alpha_k t)} - \sum_{k \neq j} \alpha_k \alpha_j \frac{f'(\alpha_k t) f'(\alpha_j t)}{f(\alpha_k t)\, f(\alpha_j t)} = -a$$

The solution of this equation yields the desired answer.

References

E. Lukacs, A characterization of the normal distribution. Annals of Mathematical Statistics, 13 (1942), 91 .

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

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